Glazer & Wark

7.1 (a) Show that the mean energy of electrons at absolute zero, $\bar{\varepsilon}$ is $3E_F/5$.

$$\bar{\varepsilon} = 2\times\int_0^{E_F} \varepsilon\, g(\varepsilon)\, d\varepsilon \;/N$$

$$= 2\times\int_0^{E_F} \frac{4\pi m V (2m)^{1/2}}{h^3}\, \varepsilon^{3/2}\, d\varepsilon \;/N$$

$$= 2\times \frac{4\pi m V (2m)^{1/2}}{h^3}\, \frac{2}{5}\, \varepsilon^{5/2} \Big|_0^{E_F} \;/N$$

$$= \frac{8\pi m V (2m)^{1/2}}{h^3}\, \frac{2}{5}\, E_F^{3/2}\, E_F \;/N$$

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \Rightarrow E_F^{3/2} = \frac{\hbar^3}{(2m)^{3/2}}\,\frac{3\pi^2 N}{V}.$$

$$\Rightarrow \bar{\varepsilon} = \cancel{\frac{8\pi m V 2^{1/2} m^{1/2}}{h^3}}\, \frac{2}{5}\, \cancel{\frac{\hbar^3}{2^3\pi^3\, 2^{3/2}\, m^{3/2}}}\, \cancel{\frac{3\pi^2 N}{V\, N}}.\, E_F$$

$$= \frac{3}{5} E_F.$$

(b). Show that the ratio of the mean-squared-speed to the square of the mean speed is 16/15.

$\overline{v^2}$ is easy, it's just $\bar{\varepsilon} = \frac{1}{2} m \overline{v^2}$

$$\Rightarrow \overline{v^2} = \frac{6}{5m} E_F.$$

For $\bar{v}^2$, we do the integral,

$$\bar{v} = 2 \times \int_0^{E_F} \frac{g(\varepsilon)}{N} \sqrt{\frac{2\varepsilon}{m}}\, d\varepsilon$$

$$= 2 \times \int_0^{E_F} \frac{4\pi m V}{N} (2m)^{1/2} \frac{2^{1/2}\varepsilon^{1/2}}{m^{1/2}} \varepsilon^{1/2}\, d\varepsilon$$

$$= 2 \times \int_0^{E_F} \frac{4\pi m V}{N} \frac{2}{h^3} \varepsilon\, d\varepsilon$$

$$= \frac{8\pi m V}{N h^3} E_F^2 .$$

using $N = \frac{2^{3/2} m^{3/2} E_F^{3/2}}{\hbar^3} \frac{V}{3\pi^2}$, we have

$$\bar{v} = \frac{8\pi m V}{h^3} E_F^{1/2} \frac{\hbar^3}{2^{3/2} m^{3/2}} \frac{3\pi^2}{V}$$

$$= \frac{E_F^{1/2}}{2^{3/2} m^{1/2}} 3 \quad \Rightarrow \quad \bar{v}^2 = \frac{9}{8m} E_F ,$$

$$\Rightarrow \frac{\overline{v^2}}{\bar{v}^2} = \frac{2\cancel{8}}{5} \frac{8}{\cancel{9}\,3} = \frac{16}{15} .$$